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Chirped self-similar solutions of a generalized nonlinear Schrödinger equation model

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Exact chirped self-similar solutions of the generalized nonlinear Schrödinger equation with varying dispersion, nonlinearity, gain or absorption, and nonlinear gain have been found. The stability of these nonlinearly chirped solutions is then demonstrated numerically by adding Gaussian white noise and by evolving from an initial chirped Gaussian pulse, respectively. It is reported that the pulse position of these chirped pulses can be precisely piloted by tailoring the dispersion profile, and that the sech-shaped solitary waves can propagate stably in the regime of $\beta(z)\gamma(z)>0$ as well as the regime of $\beta(z)\gamma(z)<0$, according to the magnitude of the nonlinear chirp parameter. Our theoretical predictions are in excellent agreement with the numerical simulations.

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Self-similarity has become a topic of growing interest in the description of many complicated phenomena, including the scaling properties of turbulent flow [1], the formation of fractals in nonlinear system [2], and the wave collapse in hydrodynamics [3] over the years. It arises after the influences of initial conditions have faded away, but the system is still far from the ultimate state. In the field of nonlinear optics, a limited number of self-similar phenomena have been reported. To name a few, the self-similar behaviors in stimulated Raman scattering [4], the evolution of self-written waveguides [5], the formation of Cantor set fractals in materials that support spatial solitons [6], and the nonlinear propagation of pulses in optical fibers [7] were investigated. Recently, this concept has been extended to an optical fiber amplifier [8] and a laser resonator [9]. In both cases, parabolic pulses were shown to propagate self-similarly, and the predicted evolution was verified experimentally.

As is well known, the presence of self-similarity implies an inherent spatial and/or temporal order that can be exploited in the mathematical treatment of the governing equations [10]. We especially note that exact self-similar solutions of the nonlinear Schrödinger (NLS) equation with distributed coefficients were found by using symmetry reduction [11]. It is remarkable that these solutions are very consistent with the solitary wave or soliton solutions presented by Serkin and Hasegawa [12], by use of another methodology. As reported, these self-similar pulses or solitary waves possess a strictly linear chirp that leads to efficient compression or amplification, and thus are particularly useful in the design of optical fiber amplifiers, optical pulse compressors, and solitary wave based communication links [8–12].

In this paper, we follow the works in Refs. [10–12] and consider the system described by the generalized NLS equation with varying dispersion, nonlinearity, gain or loss, and nonlinear gain or absorption. Under certain parametric conditions, exact chirped self-similar solutions are found for the first time. In contrast, these sech-shaped pulses exhibit explicitly a nonlinear chirp that arises from the nonlinear gain. By employing numerical simulations, we demonstrate the stability of these chirped solutions with respect to finite perturbations of the additive white noise [13] and by evolving from an initial chirped Gaussian pulse, respectively. In addition to the known properties of self-similar pulses [11], we report that the pulse position of our chirped self-similar pulses can be precisely piloted by tailoring the dispersion profile, and that the sech-shaped solitary waves can propagate stably in the regime of $\beta(z)\gamma(z)>0$ as well as the regime of $\beta(z)\gamma(z)<0$, where $\beta(z)$ and $\gamma(z)$ denote the corresponding group velocity dispersion (GVD) and nonlinearity functions.

The generalized NLS equation with distributed nonlinear gain governing the propagation of the optical field in a single-mode optical fiber can be written in the form

$$\psi_z = -i\frac{\beta(z)}{2}\psi_{\tau\tau} + i\gamma(z)|\psi|^2\psi + g(z)\psi + \chi(z)|\psi|^2\psi, \qquad (1)$$

where $\psi(z,\tau)$ is the complex envelope of the electric field in a comoving frame, z is the propagation distance, τ is the retarded time, $g(z)$ is the gain function, and $\chi(z)$ accounts for the nonlinear gain or absorption [14,15]. In the absence of the last term, this equation has exact self-similar solutions or solitonlike solutions that exhibit a linear chirp [11,12]. These self-similar pulses or solitary waves are rather stable when propagating along the distance, remaining localized and preserving their sech$(\beta(z)\gamma(z)<0)$ or tanh$(\beta(z)\gamma(z)>0)$ shape, with only a scaling of amplitude and temporal width. But here we are concerned with solutions characterized by a nonlinear chirp, resulting from the nonlinear gain.

To this end, the complex function $\psi(z,\tau)$ can be written as

$$\psi(z,\tau) = U(z,\tau)\exp\{im_0 \ln[U(z,\tau)] + i\Phi(z,\tau)\}, \qquad (2)$$

where m_0 denotes the nonlinear chirp parameter, and U and Φ are real functions of z and τ. As one might expect, the phase constraint made in ansatz (2) allows us to find some families of solutions in analytical form [15,16]. It is noteworthy that the phase Φ is assumed to be

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$$\Phi(z,\tau) = a(z) + b(z)[\tau - \tau_p(z)] + c(z)[\tau - \tau_p(z)]^2, \quad (3)$$

where the pulse position τ_p is a function of z. Subsequently, Eqs. (1)–(3) yield a self-similar form of the amplitude

$$U(z,\tau) = \frac{F(T)}{\sqrt{1-2c_0D(z)}} \exp(G(z) - m_0\Theta(z)), \quad (4)$$

where the scaling variable T is given by

$$T = \frac{\tau - \tau_p(z)}{1-2c_0D(z)}. \quad (5)$$

The other functions $D(z)$, $\Theta(z)$, $G(z)$ and $\tau_p(z)$ in Eqs. (4) and (5) take the forms

$$D(z) = \int_0^z \beta(z')dz', \quad (6)$$

$$\Theta(z) = -\lambda \int_0^z \frac{\beta(z')}{[1-2c_0D(z')]^2}dz', \quad (7)$$

$$G(z) = \frac{1}{2}\ln\left(\frac{m_0^2-2}{2}\rho(0)\right) + \int_0^z g(z')dz', \quad (8)$$

$$\tau_p(z) = \tau_c - b_0D(z), \quad (9)$$

where b_0, c_0, λ, and τ_c are the integration constants, and $\rho(z)=\beta(z)/\gamma(z)$. In terms of the above functions (6)–(8), the phase parameters $a(z)$, $b(z)$, and $c(z)$ in Eq. (3) are found to be

$$a(z) = a_0 + \frac{1+m_0^2}{2}\Theta(z) - \frac{b_0^2}{2}D(z) - \frac{m_0}{2}\ln[|c(z)|] - m_0G(z), \quad (10)$$

$$b(z) = b_0, \quad (11)$$

$$c(z) = \frac{c_0}{1-2c_0D(z)}. \quad (12)$$

It should be emphasized that the existence of such self-similar solutions is conditional on the following two formulas:

$$g(z) = \frac{1}{2\rho(z)}\frac{d}{dz}\rho(z) + \frac{c_0\beta(z)}{1-2c_0D(z)} - \frac{\mu\lambda m_0\beta(z)}{[1-2c_0D(z)]^2}, \quad (13)$$

$$\frac{\chi(z)}{\gamma(z)} = \frac{3m_0}{m_0^2-2}, \quad (14)$$

where $\mu=1$ or $\frac{3}{4}$. The former condition describes that the four parameter functions in Eq. (1) cannot be chosen independently. The latter implies that the nonlinear chirp parameter m_0 is in fact determined by the ratio $\chi(z)/\gamma(z)$. In our analytical work, it requires that the ratio is a constant. From the physical point of view, we come to the conclusion that $m_0^2 \neq 2$ for arbitrary nonlinear materials.

As a result, for $\mu=1$, the function $F(T)$ in Eq. (4) can be determined by solving the nonlinear differential equation

$$\frac{d^2F}{dT^2} - \lambda F + 2F^3 = 0, \quad (15)$$

where $dF/dT \neq 0$. Then it follows from Eqs. (4)–(15) for the case $\lambda = \tau_0^{-2}$ that

$$U(z,\tau) = \frac{\sqrt{(m_0^2-2)\rho(z)}}{\tau_0\sqrt{2}[1-2c_0D(z)]}\,\mathrm{sech}\left(\frac{\tau-\tau_c+b_0D(z)}{\tau_0[1-2c_0D(z)]}\right), \quad (16)$$

where $(m_0^2-2)\rho(z)>0$, and τ_0 is the initial pulse width. We note that the analog of the so-called kink solitary wave cannot exist because of the constraint $U(z,\tau)>0$ made in ansatz (2) for nonlinear chirp ($m_0\neq 0$). But, for appropriate constant $\lambda=(2-\kappa^2)/\tau_0^2$, there exist two bounded periodic solutions which are proportional to Jacobian elliptic functions $\mathrm{dn}(T/\tau_0,\kappa)$ and $\mathrm{nd}(T/\tau_0,\kappa)$, where κ is an arbitrary parameter in the interval $0<\kappa<1$. On the other hand, if $\mu=\frac{3}{4}$, we can obtain readily from Eqs. (4)–(14) the homogeneous solution (independent of τ) for arbitrary $\lambda(\neq 0)$,

$$U(z,\tau) = \frac{\sqrt{\lambda(m_0^2-2)\rho(z)}}{2[1-2c_0D(z)]}, \quad (17)$$

where $\lambda(m_0^2-2)\rho(z)>0$.

It is further shown that when $m_0=0$ (linear chirp), our analytical solutions (16) and (17) remain valid. But, considering the fact that the constraint in ansatz (2) vanishes, the tanh-shaped solitary wave solution and other cnoidal wave solutions of Eq. (15) come into existence with appropriate constants λ [11]. By the same token, Eq. (17) also becomes valid for $\lambda=0$. On the other hand, we note that the trivial solution $\psi(z,\tau)=0$ is always in existence, independent of all related parameters. If $m_0=0$, such a trivial solution can follow easily from Eq. (17) with $\lambda=0$. If $m_0\neq 0$, we cannot obtain this trivial solution from Eq. (16) or (17) directly, but we have $U(z,\tau)\to 0$, as $m_0^2\to 2$. It states that the trivial solution is the limit case of our nonlinearly chirped solutions.

Next, we wish to cite an example illustrative of some fascinating features of our chirped solution (16) by considering the system in which the GVD and the nonlinearity are distributed according to [12]

$$\beta(z) = \beta_0\cos(\sigma z), \quad \gamma(z) = \gamma_0\cos(\sigma z), \quad (18)$$

where β_0, γ_0, and $\sigma(\neq 0)$ are arbitrary constants. In this instance, the corresponding gain and nonlinear gain functions given by Eqs. (13) and (14) read

$$g(z) = \frac{\sigma\nu\cos(\sigma z)}{2-2\nu\sin(\sigma z)} - \frac{m_0\beta_0\cos(\sigma z)}{\tau_0^2[1-\nu\sin(\sigma z)]^2}, \quad (19)$$

$$\chi(z) = \frac{3m_0\gamma_0}{m_0^2-2}\cos(\sigma z), \quad (20)$$

where the parameter $\nu=2c_0\beta_0/\sigma$ ($|\nu|<1$) has been introduced for brevity. Hence the amplitude of the solitary wave solution given by Eq. (16) reduces to

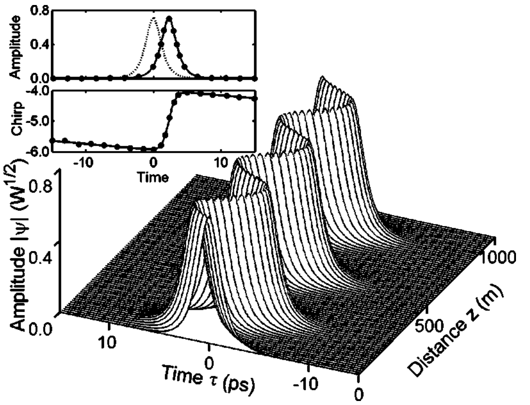

FIG. 1. Evolution of an initial nonlinearly chirped pulse $\psi(0,\tau)=[(1/\sqrt{2})\mathrm{sech}(\tau)]^{1+\mathrm{i}}\exp\,(\mathrm{i}5\tau+\mathrm{i}0.01\tau^2)$ in the regime of $\beta(z)\gamma(z)<0$. The insets compare our analytical results (21) (in unit of $\mathrm{W}^{1/2}$) and (22) (in unit of THz) at $z=1000$ m (solid line) with the numerical simulations (circles) as well as with the initial distribution in amplitude (dotted line).

$$U(z,\tau)=\frac{\sqrt{(m_0^2-2)\rho(0)}}{\sqrt{2}W(z)}\,\mathrm{sech}\left(\frac{\tau-\tau_p}{W(z)}\right), \qquad (21)$$

where $\rho(0)=\beta_0/\gamma_0$, $W(z)=\tau_0[1-\nu\sin(\sigma z)]$, and the pulse position τ_p varies with $\tau_p=\tau_c-(b_0\beta_0/\sigma)\sin(\sigma z)$ that has a period $2\pi/\sigma$ in distance. The resultant chirp consisting of linear and nonlinear contributions can be derived as [8]

$$\delta\omega(\tau)=\frac{m_0}{W(z)}\tanh\left(\frac{\tau-\tau_p}{W(z)}\right)-b_0-\frac{2c_0\tau_0}{W(z)}(\tau-\tau_p). \quad (22)$$

As seen, the first term in Eq. (22) denotes the nonlinear chirp that results from the nonlinear gain, while the last two terms account for the linear chirp. It is obvious that the phase chirp (at $b_0=0$) is an odd function of $(\tau-\tau_p)/W(z)$, which varies monotonically ($m_0c_0<0$) or nonmonotonically ($m_0c_0>0$), depending on the initial combinations of m_0 and c_0.

These analytical predictions have been confirmed by numerical simulations of the underlying equation (1) by using the split-step Fourier code [13]. Figure 1 shows the evolution of an initial pulse along the fiber with the distributed parameters given by Eqs. (18)–(20). The insets compare our analytical results (21) and (22) with the numerical simulations. In these simulations, we have consider the typical situation $\beta(z)\gamma(z)<0$. The amplifier parameters are therefore given by $\beta_0=-0.01$ ps^2/m and $\gamma_0=0.01$ $\mathrm{W}^{-1}/\mathrm{m}$. The input pulse has an initial width $\tau_0=1$ ps and energy of 1 pJ. The other initial parameters used are $m_0=1$, $b_0=5$, $c_0=0.01$, and $\sigma=0.02$. It is remarkable that the pulse position of our chirped self-similar pulses varies periodically as has been expected, and our analytical results (21) and (22) are in excellent agreement with the numerical simulations, as stated in insets. Moreover, it is noted that the chirp varies nonmonotonically with τ, as $m_0c_0>0$. We have also performed numerical simulations to demonstrate the stability of these nonlinearly chirped pulses by adding Gaussian white noise [13] and by evolving from an initial chirped Gaussian pulse, respectively, as shown in

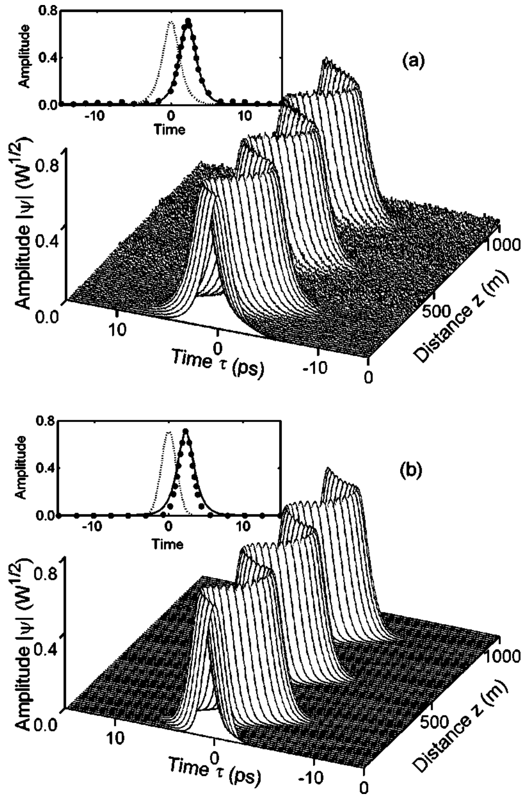

FIG. 2. Evolution of an initial pulse that is (a) the same as in Fig. 1 except for considering the finite perturbations of the additive white noise with noise intensity of 1.0×10^{-8} W/m^2, and (b) a chirped Gaussian pulse $\psi(0,\tau)=(1/\sqrt{2})\exp\,(-0.5\tau^2)$ $\times\exp\,\{i\ln[(1/\sqrt{2})\,\mathrm{sech}\,(\tau)]+i5\tau+i0.01\tau^2\}$, in the regime of $\beta(z)\gamma(z)<0$. The insets compare the result (21) with the numerical simulations, the same as in Fig. 1.

Figs. 2(a) and 2(b). It is found that these chirped pulses are rather stable and propagate self-similarly.

We proceed now to consider another situation that describes $\beta(z)\gamma(z)>0$. From Eq. (16) it is readily concluded that $m_0^2>2$. Therefore, for our present purposes, the initial parameters are given by $m_0=-2$, $b_0=0$, $c_0=0.05$, $\tau_0=1$ ps, $\beta_0=0.01$ ps^2/m, $\gamma_0=0.01$ $\mathrm{W}^{-1}/\mathrm{m}$, and $\sigma=0.005$. We have found that the hyperbolic secant pulses can propagate stably in such a regime as well (see Fig. 3). This scenario contrasts sharply with that reported in Ref. [11], where only a kink-type solitary wave is maintained in this regime. It is also different from that observed in Ref. [14], where the spatiotemporal bright soliton behaves instably in a normally dispersive planar waveguide. This newly found phenomenon would have important implications in the study of solitary wave propagations in optical fibers.

Theoretically, by constructing the appropriate profiles of gain $g(z)$ and nonlinear gain $\chi(z)$, a solitary wave can be made even smaller (larger), but well-maintained in intensity, than desired when propagating in a nonlinear medium with given dispersion and nonlinearity, corresponding to making m_0^2 close to (far from) 2. These predictions have been confirmed by our numerical simulations. Yet, experimentally, to date, it might not be easy to maintain such nonlinearly chirped pulses due to the possible technical problems, e.g.,

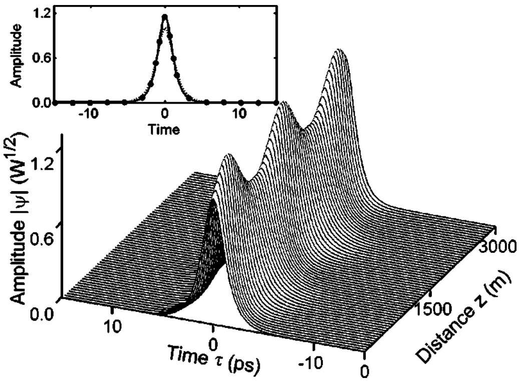

FIG. 3. Evolution of an initial chirped pulse $\psi(0,\tau) = [\mathrm{sech}(\tau)]^{1-2i}\exp(i0.05\tau^2)$ in the regime of $\beta(z)\gamma(z)>0$. The inset compares our analytical result (21) at $z=3000$ m (solid line) with the numerical simulation (circles) as well as with the initial distribution (dotted line).

the profiles of $g(z)$ and $\chi(z)$ are in general too complicated to be correctly engineered in practice. This is a subject of future investigation.

In addition, by simulating the system that involves exponentially distributed dispersion and nonlinearity, it is clearly seen that our nonlinearly chirped pulses can be compressed or amplified under certain parametric conditions, as stated in Ref. [11]. The other simulations indicate the sensitivity of these chirped pulses to parameter m_0. For example, the evolution becomes less stable as m_0^2 becomes far larger than 2, and the negative value of m_0 is more immune from the perturbations than its positive value of the same absolute magnitude, with everything else unchanged.

In conclusion, we would like to point out that our analytical results are a natural but significant generalization of those made in Refs. [10–12], by considering the nonlinear gain. As such these results can readily be applicable to compressing or spreading solitary pulses that maintain a linear chirp. Besides, our results exhibit two other important features. First, the pulse position of these chirped pulses can be precisely piloted by tailoring the dispersion profile $\beta(z)$. The second and more interesting aspect lies in the possibilities of bright solitary wave propagations in the regime of $\beta(z)\gamma(z)>0$, according to the magnitude of parameter m_0. These analytical findings suggest potential applications in areas such as optical fiber compressors, optical fiber amplifiers, nonlinear optical switches, and optical communications.

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